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Equations of Motion for the Perturbed Restricted Three-Body Problem

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BROWN and Shook[1] discussed a particular case of the perturbed restricted problem in their treatment of the motion of the Trojan asteroids. But they did not give a general method for the perturbed restricted problem. Farquhar[2] has given such general equations of motion. In the unperturbed circular problem, one normalizes to unity the (constant) distance between primaries r, their mean motion $\dot{\theta}$, and the sum of their masses. The smaller primary is of mass μ. Also, the coordinate system (x, y, z) rotates with angular velocity $\dot{\theta}$, such that the locations of masses μ, $1-\mu$ are, respectively, $x_1 = -(1-\mu)$, $x_2 = \mu$. To incorporate the indirect effect in the perturbed case, Farquhar has

$$r = 1+\rho(t), \qquad \dot{\theta} = 1+v(t) \tag{1}$$

so that the primaries are at $x_1 = -(1-\mu)(1+\rho)$, $x_2 = \mu(1+\rho)$. Then, the equations of motion are given relative to the barycenter; the independent variable is the mean anomaly l

$$\begin{aligned}
&\ddot{x}-2(1+v)\dot{y}-\dot{v}y-(1+v)^2x+\frac{1-\mu}{r_1{}^3}[x-\mu(1-\rho)]+\\
&\qquad\qquad\frac{\mu}{r_2{}^3}[x+(1-\mu)(1+\rho)] = V_x\\
&\ddot{y}+2(1+v)\dot{x}+\dot{v}x-(1+v)^2y+y\left(\frac{1-\mu}{r_1{}^3}+\frac{\mu}{r_2{}^3}\right) = V_y\\
&\ddot{z}+z\left(\frac{1-\mu}{r_1{}^3}+\frac{\mu}{r_2{}^3}\right) = V_z
\end{aligned} \tag{2}$$

where $r_1{}^2 = [x-\mu(1+\rho)]^2+y^2+z^2$, $r_2{}^2 = [x+(1-\mu)(1+\rho)]^2+y^2+z^2$, and V_x, V_y, V_z are components of perturbing acceleration on the third body (direct effect).

The indirect effect then is incorporated through the perturbation quantities ρ, v, by means of the equations

$$\ddot{r}-r\dot{\theta}^2 = -r^{-2}+\partial R/\partial r; \qquad r\ddot{\theta}+2\dot{r}\dot{\theta} = (1/r)\partial R/\partial\theta \tag{3}$$

where R is the disturbing function on the motion of the primaries. Thus, it is necessary to integrate Eqs. (3), i.e., to solve analytically a perturbed two-body problem, in order to exhibit complete equations of motion for the perturbed problem. Moreover, even in the absence of perturbations, the effects of eccentric orbits of the primaries appear in Eqs. (2) as if they were perturbations. Thus, it is of interest whether the solution of Eqs. (3) can be avoided, in deriving equations of motion.

An alternate approach follows from the derivation of equations of motion for the elliptic restricted problem by means of the Nechvile transformation.[3] In this derivation, the equations of motion in siderial coordinates are transformed and exhibit explicitly the left-hand sides of Eqs. (3) [see Eq. (61), p. 592 of Ref. 3]. Thus, let coordinates ξ, η, ζ be normalized with respect to variable r, f = true anomaly is the independent variable, $\tilde{r}_j{}^2 = (\xi-\xi_j)^2+\eta^2+\zeta^2$, $j = 1, 2$, and $\xi_1 = -(1-\mu)$, $\xi_2 = \mu$. Also, α is defined by

$$\begin{aligned}
r(df/dl)^2 &= (1+\alpha)(1+e\cos f)/r^2 = (1+e\cos f)/r^2+\\
&\quad 2(1+e\cos f)^{1/2}[\textstyle\int(\partial R/\partial\theta)\,dl]/r^{5/2}+[\textstyle\int(\partial R/\partial\theta)\,dl]^2/r^3
\end{aligned}$$

and the equations then are given

$$\begin{aligned}
&\xi''-2\eta'-[(1+\alpha)(1+e\cos f)]^{-1}\left[\xi\left(1-r^2\frac{\partial R}{\partial r}\right)-\right.\\
&\qquad\left.\frac{(1-\mu)(\xi-\mu)}{\tilde{r}_1{}^3}-\frac{\mu(\xi+1-\mu)}{\tilde{r}_2{}^3}-(\xi'+\eta)r\frac{\partial R}{\partial\theta}\right] =\\
&\qquad\qquad V_\xi\cdot r^2[(1+\alpha)(1+e\cos f)]^{-1}\\
&\eta''+2\xi'-[(1+\alpha)(1+e\cos f)]^{-1}\left[\eta\left(1-r^2\frac{\partial R}{\partial r}-\frac{1-\mu}{\tilde{r}_1{}^3}-\frac{\mu}{\tilde{r}_2{}^3}\right)-\right.\\
&\qquad\left.(\eta'-\xi)r\frac{\partial R}{\partial\theta}\right] = V_\eta\cdot r^2[(1+\alpha)(1+e\cos f)]^{-1}\\
&\zeta''+\zeta\left[1-[(1+\alpha)(1+e\cos f)]^{-1}\left(1-r^2\frac{\partial R}{\partial r}-\frac{1-\mu}{\tilde{r}_1{}^3}-\frac{\mu}{\tilde{r}_2{}^3}\right)\right]+\\
&\qquad\zeta' r\frac{\partial R}{\partial\theta} = V_\zeta\cdot r^2[(1+\alpha)(1+e\cos f)]^{-1}
\end{aligned} \tag{4}$$

In Eqs. (4), f and r appear explicitly, as in Eq. (2). Hence, for an exact treatment, one must again solve Eqs. (3), in order to obtain complete equations of motion. In such a situation, Eqs. (2) are simpler and hence preferable to Eqs. (4). But in Eqs. (4), wherever f and r appear, they are multiplied by perturbation quantities involving e, R, or V. Hence, in an approximate treatment wherein one is willing to neglect terms of order R^2, eR, etc., f and r can be given by the usual two-body relations and solution of Eqs. (3) is not required. Also α can then be neglected. Then, in such a treatment, one can use the disturbing function directly, avoiding the need for a perturbed two-body solution in deriving equations of motion.

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Index category: Lunar and Interplanetary Trajectories.

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Wall Shear in Strongly Retarded and Separated Compressible Turbulent Boundary Layers

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Nomenclature

A^+ = measure of sublayer thickness
$\bar{p}$ = mean static pressure
P^+ = dimensionless pressure gradient, $(\nu/\rho u_\tau{}^3)\,d\bar{p}/dx$

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$\bar{u}$ = mean velocity in x direction
u_τ = $(\tau/\rho)^{1/2}$
x = coordinate along surface
x_i = shock impingement point for inviscid flow
y = coordinate normal to surface
y^+ = characteristic normal Reynolds number, $[y(\tau/\rho)^{1/2}]/\nu$
K = von Kármán constant, 0.41
α_g = shock generator incidence angle
δ_0 = initial boundary-layer thickness
$\bar{\mu}$ = mean viscosity
ν = kinematic viscosity, $\bar{\mu}/\bar{\rho}$
$\bar{\rho}$ = mean density
τ = local shear, including laminar and turbulent contributions

Introduction

ONE of the principal inhibitions to the further development of models of compressible turbulent flows is the dearth of experimental data which are both reliable and complete. A parameter which is especially important in both turbulence modeling and design engineering is the distribution of wall shear stress. Unfortunately, this is also a parameter which is most difficult to measure accurately in any but the simplest flow configuration. For example, direct-measuring floating-element gages require correction for the buoyancy effects of streamwise pressure gradients. Preston tubes require extensive calibrations that vary with tube height and Reynolds number obtained under controlled pressure gradients at supersonic speeds. Finally, momentum integral methods rely on determination of small differences between large terms, a condition not conducive to good accuracy.

The present Note proposes a method by which the wall shear in compressible turbulent boundary layers, including those experiencing large adverse pressure gradients, may be obtained directly from the measured velocity and temperature profiles. The method is somewhat similar to that of Libby and Baronti[1] and its extensions,[2] but does not rely upon specific transformations of compressible to incompressible coordinates and accounts for streamwise pressure gradients. The method is applied to two strongly retarded turbulent flows at about $3 \leqq M \leqq 4$, including one flow with a substantial region of separation. The results so obtained are consistent with other data defining separation, with Preston tube measurements in regions of zero-pressure-gradient flow, and generally behave in a physically plausible manner.

Method

The present method is based on certain assumptions with regard to the flow in the immediate vicinity of the wall. First we assume that the mean flow convective terms are negligible in this vicinity, and, in addition, that the Reynolds normal stresses are negligible compared with the static pressure. This allows the following integrated form of the momentum equation to be written

$$\tau(y) = \tau_w + \int_0^y \frac{d\bar{p}}{dx}\,dy = \tau_w + \frac{d\bar{p}}{dx}y \tag{1}$$

This assumption is supported by an analysis[3] of a set of data[4] for a high-speed turbulent boundary layer subjected to a severe adverse pressure gradient. The analysis showed that the convective terms contributed less than 10% to the change in local shear stress for values of $u/u_e < 0.6$, under conditions of both zero and severe adverse pressure gradient. Further, comparisons between two calculations of velocity profiles which utilized the same turbulence model, one using the complete boundary-layer equations and the second using the small convection assumption, verified that, for $u/u_e < 0.6$, the convective terms can be neglected even in the presence of adverse pressure gradients leading to separation. For zero pressure gradient, the velocity ratio for which the assumption is valid increases to $u/u_e \approx 0.8$.

Additionally, a relationship must be assumed between the total shear- and the mean-velocity profile in the wall region. In the present work, this relation is taken to be the usual mixing-length formulation as modified by van Driest[5]:

$$\tau = \bar{\mu}\frac{\partial \bar{u}}{\partial y} + \bar{\rho}K^2y^2(1-e^{-F})^2\left|\frac{\partial \bar{u}}{\partial y}\right|\frac{\partial \bar{u}}{\partial y} \tag{2}$$

where the term $(1-e^{-F})$ is van Driest's damping factor and

$$F = y(\tau/\rho)^{1/2}/A^+\nu = y^+/A^+$$

Equation (2) is an equilibrium turbulence model which should give reasonable results in close proximity to the wall. The values chosen for the constants were $K = 0.41$ and $A^+ = 26$. The value of K has been substantiated from the data of Ref. 4 in pressure gradient flows; and the assumption of $A^+ = 26$ and the use of local properties in the definition of y^+ will be discussed subsequently. While other turbulence models could have been employed, the assumption of turbulence equilibrium in the wall region must be maintained, and with this constraint, it is anticipated that differences thus introduced would be small.

Solving Eq. (2) for $\partial\bar{u}/\partial y$ when $\tau > 0$ gives

$$\frac{\partial \bar{u}}{\partial y} = \frac{[\bar{\mu}^2 + 4\bar{\rho}K^2y^2(1-e^{-F})^2\tau]^{1/2} - \bar{\mu}}{2\bar{\rho}K^2y^2(1-e^{-F})^2} \tag{3}$$

Substituting for τ from Eq. (1) yields an equation for $\partial\bar{u}/\partial y$ in terms of $\bar{\rho}m\ \bar{u}$, y, and τ_w, in which only τ_w is unknown provided temperature profiles are available. Straightforward integration away from the wall for different choices of τ_w produces a family of velocity profiles which can be compared directly to the experimental velocity profile.

For $\tau < 0$, Eq. (2) becomes

$$\frac{\partial \bar{u}}{\partial y} = \frac{\tau}{\bar{\mu} + \bar{\rho}K^2y^2(1-e^{-F})^2|\partial\bar{u}/\partial y|} \tag{4}$$

and, again substituting Eq. (1) and integrating iteratively for different assumed values of τ_w provides a family of velocity profiles to be compared with experimental data.

Results

Two sets of experimental results have been considered: the data of Reda and Murphy[8] were taken at Mach 3 along a wall of a test section that was rectangular; the data of Rose[4] were taken at Mach 4 along a wall of a test section that was axisymmetric. In both cases, the tunnel wall was essentially adiabatic and temperature profiles were deduced from the isoenergetic energy equation. In both of these experiments, shock generators were used to induce severe adverse pressure gradients on the boundary layers on the test section walls. A wedge at angle α to the stream was used in the rectangular facility; a cone of 10° half-angle was used in the axisymmetric experiment.

An illustration of the application of the method is given in Fig. 1. The velocity profile data from Ref. 8, taken in a zero pressure gradient region, were used in this illustration. Three guesses on τ_w were employed to obtain an acceptable fit to the data. The value $\tau_w = 170$ N/m² fits the logarithmic portion of the velocity

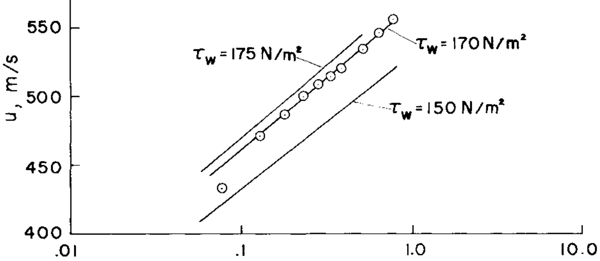

Fig. 1 Comparison of present method with the velocity distribution in zero-pressure-gradient region from the data of Ref. 8. a) Skin-friction coefficient; b) surface static-pressure distribution.

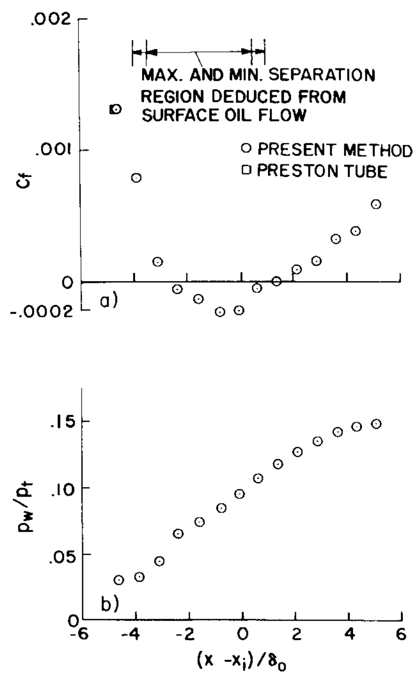

Fig. 2 Distribution of skin-friction and surface static pressure for the data of Ref. 8, $\alpha_g = 13°$. a) Skin-friction coefficient; b) surface static-pressure distribution.

profile very well, and is chosen as the correct wall value for the case shown.

A sequence of velocity profile data from Ref. 8 ($\alpha = 13°$), including those within a separated region of the flow, were examined in the same manner. The resulting distribution of skin-friction coefficient is shown in Fig. 2a. The corresponding measured surface pressures are shown in Fig. 2b. The initial skin-friction coefficient obtained by the present method is in excellent agreement with the value obtained using a Preston tube,[8] and the inferred separation and reattachment points are in reasonable agreement with those deduced from oil-flow studies.

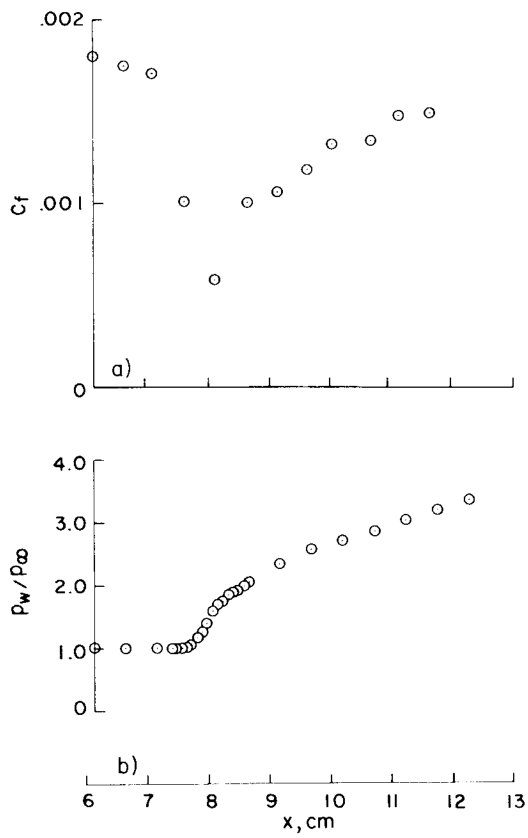

Fig. 3 Distribution of skin-friction and surface static pressure for the data of Ref. 4.

The data of Ref. 4 were analyzed by the present method to obtain the distribution of skin-friction coefficient shown in Fig. 3a. The surface-pressure distribution for this flow is shown in Fig. 3b. The skin-friction values indicate that the boundary layer was not separated, in agreement with the observation in Ref. 4.

As noted previously, the local shear stress, local properties and a constant value of $A^+ = 26$ were assumed in the evaluation of the van Driest damping factor. The local shear stress was used to avoid a singularity in Eq. (2) if $\tau_w = 0$. The local properties were used following Cebici.[7] Whether or not A^+ can rationally be taken as constant has been investigated.[6,7,9] For favorable pressure gradients, A^+ is a strong function of P^+, the nondimensional pressure gradient, while for adverse pressure gradients A^+ is less dependent on P^+. In the present study in order to determine the effect that the preceding assumptions might have on the calculated value of τ_w, a sensitivity study was performed for various relationships of the form $A^+ = A^+(P^+)$. Both the Cebici[7] and Anderson[9] relationships were considered and it was found that a variation of 5–10% of the local shear stress could be obtained. However, for the reattachment region of the flow of Ref. 8, the large values of $\partial\bar{P}/dx$ coupled with the small values of $|u_\tau|$ resulted in extremely large (≈ 1.0) values of P^+. In this case, it was found that variations of up to 100% of the local shear stress could be obtained. The redeeming feature, from the design viewpoint, is that the large local percentage uncertainty occurs only when τ_w is small so that, as a percentage of initial skin friction, the uncertainty is nowhere in excess of $\pm 10\%$ even for these rather pathological cases.

In conclusion, a simple method for obtaining the wall shear-stress from mean-velocity and temperature data in retarded and separated compressible turbulent boundary layers has been proposed. The method is easy to apply and no specific assumptions regarding compressibility transformations need be made. The results of the method appear reasonable and are consistent with the available data. Final verification of the method, however, will require additional comparisons with experimental data over a wide range of test conditions for which both profile and direct skin-friction data have been obtained.

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